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Gaussian Neighborhood Prime Decomposition of Graphs

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Abstract

A decomposition of a graph G is a collection ψ_{GNP} of edge disjoint subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each H_i is a Gaussian neighborhood prime graphs then ψ_{GNP} is called a Gaussian neighborhood prime decomposition of G. The minimum cardinality of a Gaussian neighborhood prime decomposition of G is called a Gaussian neighborhood prime decomposition number of G and it is denoted by $\pi_{GNP}(G)$. In this paper, we investigate Gaussian neighborhood prime decomposition of helm of fan graph $H_m(F_n)$, helm of wheel graph $H_m(W_n)$, sunflower planar graph SF_n.

Key words: Helm of fan graph, Helm of wheel graph, Sunflower planar graph, Neighborhood and Decomposition.

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1. Introduction

Gaussian prime labeling was introduced by Steven Klee et al [10]. In 2018, the concept of Gaussian neighborhood prime labeling was introduced by Rajesh Kumar and Mathew Varkey with respect to spiral order[5]. The Gaussian integers are 1, 1+i, 2+i, 2, 3, 3+i, 3+2i, 2+2i, 1+ 2i, 1 + 3i, 2 + 3i, 3 + 3i, 4 + 3i, 4 + 2i, 4 + i, 4, 5, 5 + i,…. In this sequel, we investigate Gaussian neighborhood prime decomposition of helm of fan graph $H_m(F_n)$, helm of wheel graph $H_m(W_n)$, sunflower planar graph SF_n.

Definition 1.1 Let $G=(V(G),E(G))$ be a graph with n vertices. A bijective function $\phi^+:V(G)\rightarrow\{\gamma_1,\gamma_2,\gamma_3,...,\gamma_n\}$ is called Gaussian neighborhood prime labeling, if the Gaussian integers in the set for every vertex $v\in V(G)$ with deg$(v)>1$, gcd$\{\phi^+(u):u\in N(v)\}=1$. A graph which admits Gaussian neighborhood prime labeling is called a Gaussian neighborhood prime graph.

Definition 1.2 A decomposition of a graph G is a collection $\psi_p=\{H_1,H_2,\ldots,H_r\}$ such that every edge of G belongs to exactly one H_i. If each H_i is a prime graph, then ψ_p is called a prime decomposition of G. The minimum cardinality of a prime decomposition of G is called the prime decomposition number of G and is denoted by $\pi_p(G)$.

Definition 1.3 [4] The helm graph H_n is obtained from a wheel graph W_n by attaching a pendant edge to each vertex of the n cycle.

Definition 1.4 [4] The fan graph F_n is obtained from the path P_n by joining all vertices of P_n to a new vertex called the center. It has $n+1$ vertices and $2n-1$ edges. That is $F_n=P_n+K_1$.

Definition 1.5[3] A wheel graph W_n is obtained by joining a single vertex to all vertices of an n cycle. In other words the wheel W_n is defined to be the join of C_n+K_1.

Definition 1.6 [6] A sunflower planar graph SF_n is obtained by a wheel graph having vertices $v_0,v_1,...,v_m$ (v_0 is essential vertex) whereas $v_1,v_2,...,v_m$ are rim vertices and other vertices $u_1,u_2,...,u_m$ such that u_j is joined to v_j and v_{j+1}.

2. Main Results

Theorem 2.1 The decomposition of helm of fan graph $H_m(F_n)$ is Gaussian neighborhood prime for all m, n $\geq$ 3.

Proof. Let $H_m(F_n)$ be the helm of fan graph with vertex set

$V[H_m(F_n)] = \{u_0\} \cup \{u_i, v_\mathrm{i} / 1 \le \mathrm{i} \le \mathrm{m}\} \cup \{w_{ij} / 1 \le \mathrm{i} \le \mathrm{m}, 1 \le \mathrm{j} \le \mathrm{n}\}$ and edge set

$E[H_m(F_n)] = \{u_0 u_\mathrm{i} / 1 \le \mathrm{i} \le \mathrm{m}\} \cup \{u_i v_i / 1 \le i \le m\} \cup \{u_i u_{\mathrm{i}+1} / 1 \le \mathrm{i} \le \mathrm{m} - 1\} \cup \{u_m u_1\} \cup$
$\{v_i w_{ij} / 1 \le i \le m, 1 \le j \le n\} \cup \{w_{ij} w_{i(j+1)} / 1 \le i \le m, 1 \le j \le n-1\}.$

Clearly, $|V[H_m(F_n)]| = m(n+2)+1$ and $|E[H_m(F_n)]| = 2m(n+1).$

Let $\psi_{GNP} = \{H_m, F_n, F_n, ..., F_n (m\, times)\}$ be a decomposition of $H_m(F_n)$.

Let m, n be the positive integers and d be the decomposition number.

Let $\psi_{GNP} = \begin{cases} (m+n-d-2)\ F_n\ \&\ H_m & if\ m \equiv 1\ (\mathrm{mod}\ 2), n = 3,4,\ldots \&\ d = 1,2,3,\ldots \\ (m+n-d-2)\ F_n\ \&\ H_m & if\ m \equiv 0\ (\mathrm{mod}\ 2), n = 3,4,\ldots \&\ d = 1,2,3,\ldots \end{cases}$

The decomposition of helm of fan graph $H_m(F_n)$ contains fan graphs F_n and a helm graph H_m. This implies that $\psi_{GNP} \supseteq \{H_m, F_n, F_n, ..., F_n (m\, times)\}$

That is $|\psi_{GNP}| \ge |H_m| + |F_n| + |F_n| + ... + |F_n| (m\, times)$

$\ge |H_m| + m|F_n|$

Hence $\pi_{GNP}[H_m(F_n)] \ge m+1.$

We claim that ψ_{GNP} is a Gaussian neighborhood prime decomposition of $H_m(F_n)$.

Let a be any vertex of F_n and H_m.

Case (i): Let $H_1 = F_n,\ n \ge 3.$

Let $\{v_0, v_1, v_2, ..., v_n\}$ be the vertices of H_1.

Define a function $\phi^+ : V(H_1) \to \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{n+1}\}$ by

$\phi^+(v_0) = \gamma_1$

$\phi^+(v_i) = \gamma_{i+1}, \quad 1 \le i \le n$

Let $a = v_0$ with $\deg(a) \ge 3.$ Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1.$

Let $a = \{v_i / 1 \le i \le n\}$ with $\deg(a) = 2\ (or)\ 3.$ Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1.$

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Case (ii): Let $H_2 = H_m,\ m \ge 3.$

Let $\{u_0, u_1, u_2, ..., u_m, v_1, v_2, ..., v_m\}$ be the vertices of H_2.

Define a function $\phi^{+}: V(H_2) \rightarrow \{\gamma_1, \gamma_2, ..., \gamma_{2m+1}\}$ by

$\phi^{+}(u_0) = \gamma_1$

$\phi^{+}(v_i) = \gamma_{i+1}, \quad 1 \le i \le m$

$\phi^{+}(u_i) = \gamma_{m+i+1}, \quad 1 \le i \le m$

Let $a = u_0$ with $\deg(a) \ge 3$. Then $\gcd\{\phi^{+}(w) / w \in N_V(a)\} = 1$.

Let $a = \{u_i / 1 \le i \le m\}$ with $\deg(a) = 4$. Then $\gcd\{\phi^{+}(w) / w \in N_V(a)\} = 1$.

Thus ϕ^{+} admits Gaussian neighborhood prime labeling.

Hence ψ_{GNP} is a Gaussian neighborhood prime decomposition of $H_m(F_n)$.

Therefore the decomposition of helm of fan graph $H_m(F_n)$ is Gaussian neighborhood prime for all m, n ≥ 3.

Theorem 2.2 The decomposition of helm of wheel graph $H_m(W_n)$ is Gaussian neighborhood prime for all m, n ≥ 3.

Proof. Let $H_m(W_n)$ be the helm of wheel graph with vertex set

$V[H_m(W_n)] = \{u_0\} \cup \{u_i / 1 \le \mathrm{i} \le \mathrm{m}\} \cup \{w_{ij} / 1 \le \mathrm{i} \le \mathrm{m}, 1 \le \mathrm{j} \le \mathrm{n}\} \cup \{w_i / 1 \le \mathrm{i} \le \mathrm{m}\}$ and edge set

$E[H_m(W_n)] = \{u_0 u_{\mathrm{i}} / 1 \le \mathrm{i} \le \mathrm{m}\} \cup \{u_i u_{\mathrm{i+1}} / 1 \le \mathrm{i} \le \mathrm{m-1}\} \cup \{u_m u_1\} \cup$
$\{u_i w_{i1} / 1 \le i \le m\} \cup \{w_{ij} w_{i(j+1)} / 1 \le i \le m, 1 \le j \le n\} \cup \{w_i w_{ij} / 1 \le i \le m, 1 \le j \le n\}$.

Clearly, $|V[H_m(W_n)]| = m(n+2)+1$ and $|E[H_m(W_n)]| = m(2n+3)$.

Let m, n be the positive integers and d be the decomposition number.

$$\text{Let} \quad \psi_{GNP} = \begin{cases} (m+n-d-2)\, W_n, C_m \,\&\, K_{1,m,m} & \text{if } m \equiv 1 \pmod 2, n = 3,4,\ldots \,\&\, d = 1,2,3,\ldots \\ (m+n-d-2)\, W_n, C_m \,\&\, K_{1,m,m} & \text{if } m \equiv 0 \pmod 2, n = 3,4,\ldots \,\&\, d = 1,2,3,\ldots \end{cases}$$

The decomposition of helm of wheel graph $H_m(W_n)$ contains a cycle graph C_m, double star graph $K_{1,m,m}$ and wheel graph W_n.

This implies that $\psi_{GNP} \supseteq \{C_m, K_{1,m,m}, W_n, W_n, ..., W_n (m\, times)\}$

That is $|\psi_{GNP}| \ge |C_m| + |K_{1,m,m}| + |W_n| + |W_n| + ... + |W_n| (m\, times)$

$$\ge |C_m| + |K_{1,m,m}| + m|W_n|$$

Hence $\pi_{GNP}[H_m(W_n)] \ge m+2$.

We claim that ψ_{GNP} is a Gaussian neighborhood prime decomposition of $H_m(W_n)$.

Let a be any vertex of C_m, $K_{1,m,m}$ and W_n.

Case (i): Let $H_1 = C_m$, $m \geq 3$.

Let $\{v_1, v_2, ..., v_m\}$ be the vertices of H_1.

Define a function $\phi^+ : V(H_1) \rightarrow \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n\}$ by

$\phi^+(v_i) = \gamma_i \quad 1 \leq i \leq m$

Let $a = \{v_i / 1 \leq i \leq m\}$ with $\deg(a) = 2$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Case (ii): Let $H_2 = K_{1,m,m}$, $m \geq 3$.

Let $\{u_0, u_1, u_2, ..., u_m, v_1, v_2, ..., v_m\}$ be the vertices of H_2.

Define a function $\phi^+ : V(H_2) \rightarrow \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{2m+1}\}$ by

$\phi^+(u_0) = \gamma_1$

$\phi^+(u_i) = \gamma_{2i} \quad 1 \leq i \leq m$

$\phi^+(v_i) = \gamma_{2i+1}, \quad 1 \leq i \leq m$

Let $a = u_0$ with $\deg(a) \geq 3$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Let $a = \{u_i / 1 \leq i \leq m\}$ with $\deg(a) = 2$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Case (iii): Let $H_3 = W_n$, $n \geq 3$.

Let $\{v_0, v_1, v_2, ..., v_n\}$ be the vertices of H_3.

Define a function $\phi^+ : V(H_3) \rightarrow \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{n+1}\}$ by

$\phi^+(v_0) = \gamma_1$

$\phi^+(v_i) = \gamma_{i+1}, \quad 1 \leq i \leq n$

Let $a = v_0$ with $\deg(a) \geq 3$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Let $a = \{v_i / 1 \leq i \leq n\}$ with $\deg(a) = 3$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Hence ψ_{GNP} is a Gaussian neighborhood prime decomposition of $H_m(W_n)$.

Therefore the decomposition of helm of fan graph $H_m(W_n)$ is Gaussian neighborhood prime for all m, n ≥ 3.

Theorem 2.3 The decomposition of the sunflower planar graph SF_n is Gaussian neighborhood prime for all $n \geq 3$.

Proof. Let SF_n be the sunflower planar graph with vertex set

$V[\ SF_n\ ] = \{u_0\} \cup \{u_i / 1 \leq \text{i} \leq \text{n}\ \} \cup \{v_i / 1 \leq \text{i} \leq \text{n}\}$ and edge set

$E[\ SF_n] = \{u_0u_i / 1 \leq \text{i} \leq \text{n}\ \} \cup \{u_iu_{i+1}, / 1 \leq \text{i} \leq \text{n-1}\} \cup \{u_nu_1\} \cup \{u_iv_i / 1 \leq \text{i} \leq \text{n}\ \} \cup \{v_iu_{i+1} / 1 \leq \text{i} \leq \text{n-1}\} \cup \{v_nu_1\}.$

Clearly, $\left| V[SF_n\ ] \right| = 2n+1$ and $\left| E\,[SF_n\ ] \right| = 4n.$

Let $\psi_{GNP} = \left\{ W_n,\ P_{2n+1} \right\}$ be a decomposition of SF_n.

Let n be the positive integer and d be the decomposition number.

Then $\psi_{GNP} = \begin{cases} (n-d-1)\, W_n \ \ \& \, P_{2n+1} & if\ n \equiv 1\, (\text{mod } 2),\ d = 1, 3, 5, \cdots \\ (n-d-1)\, W_n \ \ \& \, P_{2n+1} & if\ n \equiv 0\, (\text{mod } 2),\ d = 1, 3, 5, \cdots \end{cases}$

The decomposition of sunflower planar graph SF_n contains a wheel graph W_n and a path graph P_{2n+1}. This implies that $\psi_{GNP} \supseteq \{W_n,\ P_{2n+1}\}$

That is $|\psi_{GNP}| \geq |\ W_n\ | + |\ P_{2n+1}\ |$

Hence $\pi_{GNP}(SF_n) \geq 2.$

We claim that ψ_{GNP} *is a* Gaussian neighborhood prime decomposition of SF_n.

Let a be any vertex of W_n and P_{2n+1}.

Case (i): Let $H_1 = W_n,\ n \geq 3.$

Let $\{u_0, u_1, u_2, ..., u_n\}$ be the vertices of H_1.

Define a function $\phi^+ : V(H_1) \rightarrow \{\ \gamma_1, \gamma_2, \gamma_3, ..., \gamma_{n+1}\ \}$ by

$\phi^+(u_0) = \gamma_1$

$\phi^+(u_i) = \gamma_{i+1}, \quad 1 \leq i \leq n$

Let $a = u_0$ with $\deg(a) \geq 3.$ Then $\gcd\{\ \phi^+(w) / w \in N_V(a)\ \} = 1.$

Let $a = \{u_i / 1 \le i \le n\}$ with $\deg(a) = 3$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Case (ii): Let $H_2 = P_{2n+1}, n \ge 3$.

Let $\{v_1, v_2, v_3, \cdots, v_{2n+1}\}$ be the vertices of H_2.

Define a function $\phi^+ : V(H_2) \to \{\gamma_1, \gamma_2, ..., \gamma_{2n+1}\}$ by

$\phi^+(v_{2i-1}) = \gamma_i$, $1 \le i \le n+1$

$\phi^+(v_{2i}) = \gamma_{n+1+i}$, $1 \le i \le n$

Let $a = \{v_i / 2 \le i \le 2n\}$ with $\deg(a) = 2$. Then $\gcd\{\phi^+(w) / w \in N_V(a)\} = 1$.

Thus ϕ^+ admits Gaussian neighborhood prime labeling.

Hence ψ_{GNP} is a Gaussian neighborhood prime decomposition of SF_n.

Therefore the decomposition of sunflower planar graph SF_n is Gaussian neighborhood prime for all $n \ge 3$.

3. Conclusion

In this paper, we investigate Gaussian neighborhood prime decomposition of helm of fan graph $H_m(F_n)$, helm of wheel graph $H_m(W_n)$, sunflower planar graph SF_n. In future we will investigate Gaussian neighborhood prime de-composition of graphs using different labelings.

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